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Quantum Dynamics of a Harmonic Oscillator in a Deformed Bath

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Abstract The dissipative quantum dynamics of a harmonic oscillator in the presence of a deformed bath is investigated. The deformed bath is modelled by a collection of deformed quantum harmonic oscillators as a generalization of Hopfield model. The transition probabilities between energy levels of the oscillator are obtained perturbatively and discussed.

Keywords Dissipation · Deformed bath · Transition probability

1 Introduction

For more than a decade a nondecreasing interest has been induced to the study of deformed Lie algebras and their applications in different branches of physics. The rich mathematical structure of these algebras has produced many important results and consequences in quantum and conformal field theories, statistical mechanics, quantum and nonlinear optics, nuclear and molecular Physics. The application of these algebras in physics became intense with the introduction in 1989 by Biedenharn [1] and Macfarlane [2], of a q-deformed Weyl-Heisenberg algebra, which is deformed quantum harmonic oscillator. One of the important problems in physics is the investigation of a system coupled to its environment which in its simplest form is the standard paradigm for quantum theory of Brownian motion [3–15]. The overwhelming success of quantum theory of Brownian motion can be seen in various areas such as quantum optics, transport processes, coherence effects and macroscopic quantum

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tunnelling, electron transfer in large molecules, thermal activation processes in chemical reaction, etc. and each one forms a large body of current literature. While the early development of quantum optics initiated in the sixties and seventies was based on density operator, semigroup, noise operator or master equation methods [5, 6] primarily within weak-coupling and Markov approximation, path integral approach to quantum Brownian motion attracted wide attention in the early eighties. In the present work we consider a quantum harmonic oscillator, which can be one mode of quantized electromagnetic field, in a medium which is described by a bosonic heat bath. The layout of the paper is as follows. In Sect. 2, we solve the Heisenberg equation of motions for a harmonic oscillator in Weisskopf-Wigner approximation. In Sect. 3, we obtain transition probabilities indicating the way energy flows between oscillator and the deformed environment.

2 Langevin Equation in a Deformed Medium

In this section we solve the Heisenberg equations of motion for a damped harmonic oscillator considering Weisskopf-Wigner approximation. These allow us to obtain the Langevin equation in the presence of a deformed medium. A quantum damped harmonic oscillator is described by the Hamiltonian

$$\begin{aligned}\hat{H}_T &= \hat{H}_0 + \hat{H}_B + \hat{H}_{int}\\ &= \hbar\omega_0 \hat{a}^\dagger \hat{a} + \sum_j \frac{\hbar\omega_j}{2}(\hat{B}_j\hat{B}_j^\dagger + \hat{B}_j^\dagger\hat{B}_j) + \sum_j \hbar(k_j\hat{B}_j\hat{a}^\dagger + k_j^*\hat{B}_j^\dagger\hat{a})\end{aligned} \tag{1}$$

The first term ($\hat{H}_0$) is the Hamiltonian of the harmonic oscillator, the second term ($\hat{H}_B$) is the Hamiltonian of the deformed medium or heat-bath which is considered as a combination of deformed harmonic oscillators described by annihilation ($\hat{B}_j$) and creation ($\hat{B}_j^\dagger$) bosonic operators which can be considered as a deformed version of Hopfield model, and the third term ($\hat{H}_{int}$) is the interaction between the oscillator and its environment. The algebra of the usual operators $\hat{a}$ and $\hat{a}^\dagger$ is the Weyl-Heisenberg algebra

$$\begin{aligned}&[\hat{N}, \hat{a}] = -\hat{a}\\ &[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger\\ &[\hat{a}, \hat{a}^\dagger] = 1\end{aligned} \tag{2}$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator and the Hamiltonian describing the oscillator is defined by

$$\hat{H} = \hbar\omega_0\left(\hat{N} + \frac{1}{2}\right) \tag{3}$$

Here we have omitted the constant term $\frac{\hbar\omega_0}{2}$ from the total Hamiltonian. For any j, the deformed operators $\hat{B}_j$, $\hat{B}_j^\dagger$ are defined by their nondeformed partners $\hat{b}_j$, $\hat{b}_j^\dagger$ respectively as follows [16]

$$\begin{aligned}\hat{B}_j &= \hat{b}_j f(\hat{N}_j) = f(\hat{N}_j + 1)\hat{b}_j\\ \hat{B}_j^\dagger &= f(\hat{N}_j)\hat{b}_j^\dagger = \hat{b}_j^\dagger f(\hat{N}_j + 1)\end{aligned} \tag{4}$$

where $\hat{N}_j = \hat{b}_j^\dagger \hat{b}_j$ and $f(\hat{N}_j)$ is the deformation operator and if for each j we set $f(\hat{N}_j) \equiv 1$, we recover the usual definition of a heat bath. The deformed bosonic operators of the medium fulfill the following deformed Weyl-Heisenberg algebra

$$\begin{aligned} &[\hat{N}_j, \hat{B}_j] = -\hat{B}_j \\ &[\hat{N}_j, \hat{B}_j^\dagger] = \hat{B}_j^\dagger \\ &[\hat{B}, \hat{B}_j^\dagger] = (\hat{N}_j + 1) f^2(\hat{N}_j + 1) - \hat{N}_j f^2(\hat{N}_j) \end{aligned} \tag{5}$$

In Heisenberg picture we have

$$\frac{d\hat{a}}{dt} = -i\omega_0 \hat{a} - i \sum_j k_j \hat{B}_j \tag{6}$$

and

$$\frac{d\hat{B}_j}{dt} = \frac{-i\hat{B}_j\hat{\Omega}_j}{2} - \frac{i\hat{\Omega}_j\hat{B}_j}{2} - \frac{ik_j^*\hat{\Omega}_j\hat{a}}{\omega_j} \tag{7}$$

where

$$\hat{\Omega}_j = [(\hat{N}_j + 1) f^2(\hat{N}_j + 1) - \hat{N}_j f^2(\hat{N}_j)]\omega_j \tag{8}$$

Since $\hat{\Omega}_j$ is an operator depending on $\hat{N}_j$ nonlinearly, the analytic solutions of (6) and (7) will be impossible for an arbitrary deformation function. Therefore we simplify the problem by replacing the operator $\hat{\Omega}_j$ with its classical value $\langle\hat{\Omega}_j\rangle$ in the presence of the bath. For this purpose let us assume that the bath has a Maxwell-Boltzmann distribution, in this case we have

$$\begin{aligned} \langle\hat{\Omega}_j\rangle &= \mathrm{tr}(\rho_B^T \hat{\Omega}_j) = \mathrm{tr}\left[\frac{1}{Z} e^{-\beta \hat{H}_B \hat{\Omega}_j} \hat{\Omega}_j\right] = \frac{1}{Z}\sum_{n_j=0}^{\infty} \langle n_j | e^{-\beta \sum_j \frac{\hbar\omega_j}{2}(\hat{B}_j\hat{B}_j^\dagger + \hat{B}_j^\dagger \hat{B}_j)} \hat{\Omega}_j | n_j\rangle \\ &= \frac{\omega_j}{Z} \sum_{n=0}^{\infty} e^{\frac{-\beta\omega_j}{2}[(n+1)f^2(n+1)+nf^2(n)]} [(n+1)f^2(n+1) + nf^2(n)] \end{aligned} \tag{9}$$

where $Z = \mathrm{tr}(e^{-\beta \hat{H}_B})$ is the partition function of the deformed bath. Now (7) becomes

$$\frac{d\hat{B}_j}{dt} = -i\langle\hat{\Omega}_j\rangle \hat{B}_j - i \frac{k_j^* \langle\hat{\Omega}_j\rangle \hat{a}}{\omega_j} \tag{10}$$

with the following solution

$$\hat{B}_j(t) = e^{-i\langle\hat{\Omega}_j\rangle t} - \frac{ik_j^*\langle\hat{\Omega}_j\rangle}{\omega_j} \int_0^t a(t') e^{i\langle\hat{\Omega}_j\rangle(t'-t)} dt' \tag{11}$$

Considering this recent solution, for (6) we find

$$\frac{d\hat{a}}{dt} = -i\omega_0\hat{a} - \sum_j \frac{|k_j|^2 \langle\hat{\Omega}_j\rangle^2}{\omega_j^2} \int_0^t a(t') e^{i\langle\hat{\Omega}_j\rangle(t'-t)} dt' + G_a \tag{12}$$

where we have defined

$$G_a = -i \sum_j \frac{k_j \langle \hat{\Omega}_j \rangle \hat{B}_j(0)}{\omega_j} e^{-i\langle \hat{\Omega}_j \rangle t} \tag{13}$$

In order to removing the high-frequency behavior from (12) let us define the new operator $\hat{A}$ as

$$\hat{A}(t) = \hat{a}(t) e^{i\omega_0 t} \tag{14}$$

and (12) reduce to

$$\frac{d\hat{A}}{dt} = -\sum_j \frac{|k_j|^2 \langle \hat{\Omega}_j \rangle^2}{\omega_j^2} \int_0^t dt' A(t') \exp[i(\langle \hat{\Omega}_j \rangle - \omega_0)(t' - t)] + G_A \tag{15}$$

where

$$G_A = -i \sum_j \frac{k_j \langle \hat{\Omega}_j \rangle \hat{B}_j(0)}{\omega_j} \exp[-i(\langle \hat{\Omega}_j \rangle - \omega_0)t] \tag{16}$$

If we take the Laplace transform of both sides of (15), we find

$$\bar{A}(s) = \frac{a(0) + \bar{G}_A(s)}{s + \sum_j \frac{|k_j^2| \langle \hat{\Omega}_j \rangle^2}{\omega_j^2 [s + i(\langle \hat{\Omega}_j \rangle - \omega_0)]}} \tag{17}$$

where

$$\bar{A}(s) = \int_0^\infty e^{-st} A(t) dt \tag{18}$$

and

$$\bar{G}_A(s) = -i \sum_j \frac{k_j \langle \hat{\Omega}_j \rangle \hat{B}_j(0)}{\omega_j [s + i(\langle \hat{\Omega}_j \rangle - \omega_0)]} \tag{19}$$

Using Weisskopf-Wigner approximation we have

$$\begin{aligned} -i \sum_j \frac{|k_j|^2 \langle \hat{\Omega}_j \rangle^2}{\omega_j^2 [(\langle \hat{\Omega}_j \rangle - \omega_0) - is]} &\to \lim_{s \to 0} -i \int \frac{g(\langle \hat{\Omega}_j \rangle) |k(\langle \hat{\Omega}_j \rangle)|^2 \langle \hat{\Omega}_j \rangle^2 d\langle \hat{\Omega}_j \rangle}{\omega_j^2 [(\langle \hat{\Omega}_j \rangle - \omega_0) - is]} \\ &= -i \int \frac{d\langle \hat{\Omega}_j \rangle g(\langle \hat{\Omega}_j \rangle) |k(\langle \hat{\Omega}_j \rangle)|^2 \langle \hat{\Omega}_j \rangle^2}{\omega_j^2} \left\{ \frac{1}{\langle \hat{\Omega}_j \rangle - \omega_0} + i\pi \delta(\langle \hat{\Omega}_j \rangle - \omega_0) \right\} \\ &= \frac{\gamma}{2} + i\Delta\omega \end{aligned} \tag{20}$$

where we have defined

$$\gamma = 2\pi g(\omega_0) |k(\omega_0)|^2 \tag{21}$$

$$\Delta\omega = -\int \frac{g(\langle \hat{\Omega}_j \rangle) |k(\langle \hat{\Omega}_j \rangle)|^2 d\langle \hat{\Omega}_j \rangle}{\langle \hat{\Omega}_j \rangle - \omega_0} \tag{22}$$

Thus (17) reduces to

$$\bar{A}(s)=\frac{\hat{a}(0)}{s+\frac{1}{2}\gamma+i\Delta\omega}-i\sum_j\frac{k_j\langle\hat{\Omega}_j\rangle\hat{B}_j(0)}{\omega_j[s+i(\langle\hat{\Omega}_j\rangle-\omega_0)][s+\frac{1}{2}\gamma+i\Delta\omega]} \tag{23}$$

Now by taking the inverse Laplace transform we obtain

$$\hat{a}(t)=u(t)\hat{a}(0)+\sum_j v_j(t)B_j(0)=e^{-i\omega_0 t}\hat{A}(t) \tag{24}$$

where

$$\begin{aligned} u(t)&=\exp-\left[\frac{1}{2}\gamma+i(\omega_0+\Delta\omega)\right]t \\ v_j(t)&=\frac{-k_j\langle\hat{\Omega}_j\rangle e^{-i\langle\hat{\Omega}_j\rangle t}[1-\exp i(\langle\hat{\Omega}_j\rangle-\omega_0-\Delta\omega)te^{-\gamma\frac{t}{2}}]}{\omega_j[\omega_0-\langle\hat{\Omega}_j\rangle+\Delta\omega-i\frac{\gamma}{2}]} \end{aligned} \tag{25}$$

If we neglect the small frequency shifts we find the Langevin equation

$$\frac{d\hat{A}}{dt}=-\frac{1}{2}\gamma\hat{A}+G_A(t) \tag{26}$$

where

$$G_A(t)=-i\sum_j\frac{k_j\langle\hat{\Omega}_j\rangle\hat{B}_j(0)}{\omega_j}e^{-i(\langle\hat{\Omega}_j\rangle-\omega_0)t} \tag{27}$$

The operator G_A is a random or noise operator and the term $-(\frac{\gamma}{2})\hat{A}$ is responsible for a drift motion. Since the noise operator G_A is a linear combination in bosonic operators $\hat{B}_j$ so its reservoir average is zero.

$$\langle G_A(t)\rangle_B=\mathrm{tr}_B(\rho_B^T G_A(t))=0 \tag{28}$$

where tr_B means taking trace over the reservoir degrees of freedom. Therefor from (26) we find

$$\frac{d}{dt}\langle\hat{A}(t)\rangle_B=-\frac{1}{2}\gamma\langle\hat{A}(t)\rangle_B \tag{29}$$

with the solution

$$\langle\hat{A}(t)\rangle=e^{-\frac{\gamma t}{2}}\hat{a}(0) \tag{30}$$

note that $\langle\hat{A}(t)\rangle_B=\langle\hat{a}(0)\rangle_B\equiv\hat{a}(0)$. The time-evolution of the harmonic oscillator number operator $\hat{a}^\dagger\hat{a}$ is found as

$$\frac{d}{dt}\hat{a}^\dagger\hat{a}=\frac{1}{i\hbar}[\hat{a}^\dagger\hat{a},\hat{H}_T]=i\sum_j k_j^*\hat{B}_j^\dagger\hat{a}-i\hat{a}^\dagger\sum_j k_j\hat{B}_j \tag{31}$$

and using (11) it reduces to

$$\frac{d}{dt}\hat{a}^\dagger\hat{a} = -\sum_j \frac{|k_i|^2\langle\hat{\Omega}_j\rangle^2}{\omega_j^2}\int_0^t [\hat{a}^\dagger(t')e^{-i\langle\hat{\Omega}_j\rangle(t'-t)}\hat{a}(t) + \hat{a}^\dagger(t)a(t')e^{i\langle\hat{\Omega}_j\rangle(t'-t)}]dt' + G_{a^\dagger a}$$
$$= -\gamma\hat{a}^\dagger(t)\hat{a}(t) + G_{a^\dagger a} \tag{32}$$

where we have used the Weisskopf-Wigner approximation and defined

$$G_{a^\dagger a} = i\sum_j \left\{\frac{k_j^*\langle\hat{\Omega}_j\rangle}{\omega_j}\hat{B}_j^\dagger(0)e^{i\langle\hat{\Omega}_j\rangle t}a(t) - i\frac{k_j\langle\hat{\Omega}_j\rangle}{\omega_j}\hat{a}^\dagger(t)\hat{B}_j(0)e^{-i\langle\hat{\Omega}_j\rangle t}\right\} \tag{33}$$

If we insert the solution (24), $G_{a^\dagger a}$ becomes

$$G_{a^\dagger a} = i\sum_j \frac{k_j^*\langle\hat{\Omega}_j\rangle}{\omega_j}e^{i(\langle\hat{\Omega}_i\rangle-\omega_\circ)t}e^{(-\frac{\gamma}{2})t}\hat{a}(0)$$
$$+\sum_{j,k}\frac{k_j^*k_k\langle\hat{\Omega}_j\rangle\langle\hat{\Omega}_k\rangle}{\omega_j\omega_k}\hat{B}_j^\dagger(0)e^{i(\langle\hat{\Omega}_j\rangle-\omega_0)t}\hat{B}_k(0)\frac{[e^{-i(\langle\hat{\Omega}_k\rangle-\omega_\circ)t} - e^{(-\frac{\gamma}{2})t}]}{(\frac{\gamma}{2}) - i(\langle\hat{\Omega}_k\rangle-\omega_\circ)}$$
$$+ h.c. \tag{34}$$

Using the equations

$$\mathrm{tr}_B(\rho_B^T\hat{B}_j^\dagger(0)) = \mathrm{tr}_B(\rho_B^T\hat{B}_j(0)) = \langle\hat{B}_j^\dagger(0)\rangle_B = \langle\hat{B}_j(0)\rangle = 0$$
$$\mathrm{tr}_B(\hat{B}_j^\dagger(0)\rho_B^T\hat{B}_k(0)) = \langle\hat{B}_j^\dagger(0)\hat{B}_k(0)\rangle_B = \delta_{jk}\bar{n}_j \tag{35}$$

where

$$\bar{n}_j = \frac{n_j f^2(n_j)e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_BT}}}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_BT}} - 1} \tag{36}$$

now we find from (34)

$$\langle G_{a^\dagger a}(t)\rangle_B$$
$$= \sum_j \frac{|k_j|^2\langle\hat{\Omega}_j\rangle^2}{\omega_j^2}\left\{\frac{1 - e^{[i(\langle\hat{\Omega}_j\rangle-\omega_\circ)-\frac{\gamma}{2}]t}}{\frac{\gamma}{2} - i(\langle\hat{\Omega}_j\rangle-\omega_\circ)} + c.c.\right\}$$
$$= \sum_j \frac{|k_j|^2\langle\hat{\Omega}_j\rangle^2\bar{n}_j\{\gamma - \gamma e^{-(\frac{\gamma}{2})t}\cos(\langle\hat{\Omega}_j\rangle-\omega_\circ)t + 2(\langle\hat{\Omega}_j\rangle-\omega_\circ)e^{(-\frac{\gamma}{2})t}\sin(\langle\hat{\Omega}_j\rangle-\omega_\circ)t\}}{\omega_j^2[(\frac{\gamma}{2})^2 + (\langle\hat{\Omega}_j\rangle-\omega_\circ)^2]} \tag{37}$$

Since $\frac{|k_j|^2\langle\hat{\Omega}_j\rangle^2}{\hat{\omega}_j^2}$ is slowly varying and the summand is so strongly peak at $\langle\hat{\Omega}_j\rangle = \omega_0$, we may convert the sum to an integral and remove the slowly varying factors. This gives

$$\langle G_{a^\dagger a}\rangle = \frac{|k(\omega_\circ)|^2\langle\hat{\Omega}(\omega_\circ)\rangle^2}{\omega_\circ^2}g(\omega_\circ)\bar{n}(\omega_\circ)\int_{-\infty}^{\infty}dx\frac{\{\gamma - \gamma e^{-\frac{\gamma}{2}t}\cos(xt) + 2xe^{(-\frac{\gamma}{2})t}\sin(xt)\}}{(\frac{\gamma}{2})^2 + x^2} \tag{38}$$

where $x = \langle\hat{\Omega}_j\rangle - \omega_\circ$ and $dx = d\langle\hat{\Omega}_j\rangle$ and we have assumed that the reservoir modes are closely spaced with $g(\omega_j)d\omega_j$ the number of modes between ω_j and $\omega_j + d\omega_j$. Using the following definite integrals

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dx}{(\frac{\gamma}{2})^2 + x^2} = \frac{2\pi}{\gamma} \\
&\int_{-\infty}^{\infty} \frac{\cos(xt)}{(\frac{\gamma}{2})^2 + x^2} dx = \frac{2\pi}{\gamma} e^{-(\frac{\gamma}{2})|t|} \\
&\int_{-\infty}^{\infty} \frac{x\sin(x|t|)}{(\frac{\gamma}{2})^2 + x^2} dx = \begin{cases} \pi e^{-(\frac{\gamma}{2})|t|} & t \neq 0 \\ 0 & t = 0 \end{cases}
\end{aligned} \tag{39}$$

Equation (38) reduces to

$$\langle G_{a^\dagger a}\rangle = \gamma\bar{n} \tag{40}$$

where we used the relation (21). Now by averaging both sides of (32) and using (40) we obtain

$$\frac{d}{dt}\langle\hat{a}^\dagger a\rangle_B = -\gamma\langle\hat{a}^\dagger a\rangle_B + \gamma\bar{n} \tag{41}$$

with the solution

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = e^{-\gamma t}\hat{a}^\dagger(0)\hat{a}(0) + \bar{n}[1 - e^{-\gamma t}]. \tag{42}$$

Now let us rewrite (32) and include the reservoir average of $G_{a^\dagger a}$, since $\hat{A}^\dagger\hat{A} = \hat{a}^\dagger\hat{a}$ we have

$$\frac{d}{dt}\hat{A}^\dagger\hat{A} = -\gamma\hat{A}^\dagger\hat{A} + \gamma\bar{n} + G_{A^\dagger A} \tag{43}$$

where

$$G_{A^\dagger A} = G_{a^\dagger a} - \langle G_{a^\dagger a}\rangle_B = G_{a^\dagger a} - \gamma\bar{n} \tag{44}$$

Note that $\langle G_{A^\dagger A}(t)\rangle_B = 0$, so the (43) leads to the same (41). The Langevin force $G_{A^\dagger A}$ has zero thermal average and the remaining terms in (43) give a thermally averaged drift.

3 Transition Probabilities

In this section we calculate the transition probability between the oscillator energy states which indicates the way energy flows between the system and reservoir. Let us write the Hamiltonian (1) as

$$\begin{aligned}
&\hat{H}_T = \hat{H}_0 + \hat{H}' \\
&\hat{H}_0 = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \sum_j \frac{\hbar\omega_j}{2}(\hat{B}_j\hat{B}_j^\dagger + \hat{B}_j^\dagger\hat{B}_j) \\
&\hat{H}' = \hbar\sum_j(k_j\hat{B}_j\hat{a}^\dagger + k_j^*\hat{B}_j^\dagger\hat{a})
\end{aligned} \tag{45}$$

In interaction picture we have

$$\hat{H}'_I(t) = e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{H}'(0) e^{\frac{-iH_0 t}{\hbar}} = \hbar \sum_j [k_j \hat{B}_{Ij} \hat{a}_I^\dagger + k_j^* \hat{B}_{Ij}^\dagger \hat{a}_I] \tag{46}$$

and for later convenience let us define new operator $\gamma(\hat{N}_j)$ by

$$\hat{B}_j \hat{B}_j^\dagger + \hat{B}_j^\dagger \hat{B}_j = \gamma(\hat{N}_j) \tag{47}$$

The time-evolution operator U_I up to the first-order time-dependent perturbation theory is given by

$$\begin{aligned} \hat{U}_I(t,0) &= 1 - \frac{i}{\hbar} \int_0^t dt_1 \hat{H}'_I(t_1) \\ &= 1 - i \int_0^t \left(\sum_j k_j \hat{B}_{Ij}(t_1) \hat{a}^\dagger(0) e^{i\omega_0 t_1} + \sum_j k_j^* \hat{B}_{Ij}^\dagger(t_1) \hat{a}(0) e^{-i\omega_0 t_1} \right) dt_1 \end{aligned} \tag{48}$$

and the density operator of the total system $\hat{\rho}_I(t)$, up to the first-order perturbation, is

$$\hat{\rho}_I(t) = \hat{U}_I(t) \hat{\rho}_I(0) \hat{U}_I^\dagger(t) \tag{49}$$

Let the initial density matrix of the total system be a product state

$$\hat{\rho}_I(0) = \hat{\rho}_s(0) \otimes \hat{\rho}_B^T \tag{50}$$

Where $\hat{\rho}_s$ is the initial density matrix of the oscillator and $\hat{\rho}_B^T$ is the initial density matrix of the deformed reservoir which we assume has a Maxwell-Boltzmann distribution. For finding the reduced density matrix of the oscillator, we use the following relations which can be found easily

$$\begin{aligned} &\mathrm{tr}_B[\hat{B}_{Ij}(t_1)\hat{\rho}_B^T \hat{B}_{Ik}(t_2)] = \mathrm{tr}_B[\hat{B}_{Ij}^\dagger(t_1)\hat{\rho}_B^T \hat{B}_{Ik}^\dagger(t_2)] = 0 \\ &\mathrm{tr}_B[\hat{B}_{Ij}(t_1)\hat{\rho}_B^T \hat{B}_{Ik}^\dagger(t_2)] = \frac{(n_j+1) f^2(n_j+1)\delta_{jk}}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} - 1} e^{\frac{i\omega_j}{2}[\gamma(n_j+1)-\gamma(n_j)][t_2-t_1]} \\ &\mathrm{tr}_B[\hat{B}_{Ij}^\dagger(t_1)\hat{\rho}_B^T \hat{B}_{Ik}(t_2)] = \frac{n_j f^2(n_j) e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} \delta_{jk}}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} - 1} e^{\frac{i\omega_j}{2}[\gamma(n_j-1)-\gamma(n_j)][t_2-t_1]} \end{aligned} \tag{51}$$

Let us for simplicity define

$$\Omega(n_j) = \frac{1}{2}[(n_j+2) f^2(n_j+2) - n_j f^2(n_j)] \tag{52}$$

then

$$\begin{aligned} \gamma(n_j+1) - \gamma(n_j) &= 2\Omega(n_j) \\ \gamma(n_j-1) - \gamma(n_j) &= -2\Omega(n_j-1) \end{aligned} \tag{53}$$

Now the reduced density matrix can be obtained by tracing out the reservoir degrees of freedom

$$\begin{aligned}\hat{\rho}_s(t) &= \mathrm{tr}_B[\hat{\rho}_I(t)]\\ &= \hat{\rho}_s(0) + \int_0^t dt_1 \int_0^t dt_2 \sum_j k_j \Bigg[\frac{(n_j+1) f^2(n_j+1)}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} - 1} e^{i[\omega_j \Omega(n_j) - \omega_\circ][t_2 - t_1]}\ \mathrm{tr}_s[\hat{a}^\dagger(0)\hat{a}(0)]\\ &\quad + \frac{n_j f^2(n_j) e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}}}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} - 1} e^{-i[\omega_j \Omega(n_j - 1) - \omega_\circ][t_2 - t_1]}\ \mathrm{tr}_s[\hat{a}(0)\hat{a}^+(0)] \Bigg] \end{aligned} \tag{54}$$

Now as an example assume that the oscillator is initially in its Nth excited state, i.e., $\hat{\rho}_s(0) = |N\rangle\langle N|$, then the matrix components of the density matrix in an arbitrary time is

$$\begin{aligned}\hat{\rho}_{s;n,m} &= \delta_{n,N}\delta_{N,m}\\ &\quad + \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_j k_j \Bigg[\frac{(n_j+1) f^2(n_j+1)}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} - 1} e^{i[\omega_j \Omega(n_j) - \omega_\circ][t_2 - t_1]}\\ &\quad \times (N+1)\delta_{n,N+1}\delta_{N,m-1}\\ &\quad + \frac{n_j f^2(n_j) e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}}}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} - 1} e^{-i[\omega_j \Omega(n_j - 1) - \omega_\circ][t_2 - t_1]} N \delta_{n,N-1}\delta_{N,m+1} \Bigg] \end{aligned} \tag{55}$$

Let us find the decay amplitude

$$\begin{aligned}\Gamma_{N\to N-1} &= \mathrm{tr}_s[|N-1\rangle\langle N-1|\hat{\rho}_s(t)] = \langle N-1|\hat{\rho}_s(t)|N-1\rangle\\ &= \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_j k_j \Bigg[\frac{N n_j f^2(n_j)}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} - 1} e^{-i[\omega_j \Omega(n_j - 1) - \omega_\circ][t_2 - t_1]} \Bigg] \end{aligned} \tag{56}$$

By changing the integration variables as

$$\begin{aligned} u &= t_2 - t_1\\ v &= t_2 + t_1\\ du\, dv &= 2\, dt_2\, dt_1 \end{aligned} \tag{57}$$

we find

$$\Gamma_{N\to N-1} = 2t \sum_j k_j \Bigg[\frac{N n_j f^2(n_j)}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}} - 1} \left(\frac{\sin[\omega_j \Omega(n_j - 1) - \omega_\circ]t}{\omega_j \Omega(n_j - 1) - \omega_\circ} \right) \Bigg] \tag{58}$$

In the large-time limit we have

$$\lim_{t\to\infty} \frac{\sin(\alpha t)}{\pi \alpha} = \delta(\alpha) \tag{59}$$

therefore the decaying rate in sufficiently large times is given by

$$\Gamma_{N\to N-1} = 2\pi t \int_0^\infty d\omega_j g(\omega_j)k(\omega_j)\left[\frac{Nn_j f^2(n_j)}{e^{\frac{\hbar n_j f^2(n_j)\omega_j}{K_B T}}-1}\delta(\omega_j\Omega(n_j-1)-\omega_0)\right]$$

$$= 2\pi t g\left(\frac{\omega_0}{\Omega(n_j-1)}\right)k\left(\frac{\omega_0}{\Omega(n_j-1)}\right)\left[\frac{Nn_j f^2(n_j)}{e^{\frac{\hbar n_j f^2(n_j)}{K_B T}[\frac{\omega_0}{\Omega(n_j-1)}]}-1}\right] \tag{60}$$

where $g(\omega_j)$ is the mode density. The rate of getting energy can also be obtained in a similar way to be

$$\Gamma_{N\to N+1} = 2\pi t g\left(\frac{\omega_\circ}{\Omega(n_j)}\right)k\left(\frac{\omega_\circ}{\Omega(n_j)}\right)\left[\frac{(N+1)(n_j+1)f^2(n_j+1)e^{\frac{\hbar n_j f^2(n_j)}{K_B T}[\frac{\omega_0}{\Omega(n_j)}]}}{e^{\frac{\hbar n_j f^2(n_j)}{K_B T}[\frac{\omega_0}{\Omega(n_j)}]}-1}\right] \tag{61}$$

When the reservoir is in its ground state that is $T \longrightarrow 0$, then from (60) it is clear that there is only a decaying rate and energy flows from the oscillator to the deformed reservoir as expected.

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